

THE CONTINUITY PROPERTIES OF COMPACT-PRESERVING FUNCTIONS

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ABSTRACT. A function $f : X \rightarrow Y$ between topological spaces is called *compact-preserving* if the image $f(K)$ of each compact subset $K \subset X$ is compact. We prove that a function $f : X \rightarrow Y$ defined on a strong Fréchet space X is compact-preserving if and only if for each point $x \in X$ there is a compact subset $K_x \subset Y$ such that for each neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there is a neighborhood $O_x \subset X$ of x such that $f(O_x) \subset O_{f(x)} \cup K_x$ and the set $K_x \setminus O_{f(x)}$ is finite. This characterization is applied to give an alternative proof of a classical characterization of continuous functions on locally connected metrizable spaces as functions that preserve compact and connected sets. Also we show that for each compact-preserving function $f : X \rightarrow Y$ defined on a (strong) Fréchet space X , the restriction $f|LI'_f$ (resp. $f|LI_f$) is continuous. Here LI_f is the set of points $x \in X$ of local infinity of f and LI'_f is the set of non-isolated points of the set LI_f . Suitable examples show that the obtained results cannot be improved.

It is known that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if f is *preserving* in the sense that for each compact subset $K \subset \mathbb{R}$ the image $f(K)$ is compact and for each connected subset $C \subset \mathbb{R}$ the image $f(C)$ is connected. The first result of this sort has appeared in 1926 [10] and later was rediscovered and generalized by many authors: [14], [8], [7], [15], [12], [9], [13], [11], [1], [4], [5].

In this paper we study what remains of the continuity of a function $f : X \rightarrow Y$ if it merely preserves compact sets. Of course such a function can be everywhere discontinuous as shown by the classical Dirichlet function $\delta_{\mathbb{Q}} : \mathbb{R} \rightarrow \{0, 1\}$ equal 1 on rationals and 0 on irrationals. In this case the compact-preserving property follows from the local finity.

We define a function $f : X \rightarrow Y$ between topological spaces to be

- *compact-preserving* if the image $f(K)$ of each compact subset $K \subset X$ is compact;
- *locally finite* at a point $x \in X$ if for some neighborhood $O_x \subset X$ of x the image $f(O_x)$ is finite;
- *locally infinite* at $x \in X$ if f is not locally finite at x ;
- *sequentially infinite* at $x \in X$ if there is a sequence $\{x_n\}_{n \in \omega}$ that converges to x and has infinite image $\{f(x_n)\}_{n \in \omega}$.

By LI_f and SI_f we denote the sets of points $x \in X$ at which the function f is locally infinite and sequentially infinite, respectively. It is clear that $SI_f \subset LI_f$. We shall show that for a compact-preserving function $f : X \rightarrow Y$ defined on a Fréchet space X the restriction $f|SI_f$ is continuous and the set SI_f contains the set LI'_f of all non-isolated points of LI_f .

We recall that a topological space X is

- *first countable* if each point $x \in X$ possesses a countable base of neighborhoods;
- *Fréchet* if for each subset $A \subset X$ and a point $a \in \bar{A}$ from its closure there is a sequence $\{a_n\}_{n \in \omega} \subset A$ that converges to a ;
- *strong Fréchet* if for any decreasing sequence A_n , $n \in \omega$, of subsets of X and a point $a \in \bigcap_{n \in \omega} \bar{A}_n$ there is a sequence of points $a_n \in A_n$, $n \in \omega$, that converges to a .
- *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n \in \omega} \subset A$ that converges to a point $a \in X \setminus A$;

These notions relate as follows:

$$\text{first countable} \Rightarrow \text{strong Fréchet} \Rightarrow \text{Fréchet} \Rightarrow \text{sequential}.$$

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By [3, 2.4.G], a topological space X is Fréchet if and only if each subspace of X is sequential. For a function $f : X \rightarrow Y$ between topological spaces and a point $x \in X$ consider the set

$$f[x] = \{y \in Y : x \in \text{cl}_X(f^{-1}(y))\} = \bigcap \{f(O_x) : O_x \text{ is a neighborhood of } x \text{ in } X\},$$

which can be interpreted as the oscillation of f at x . If f is continuous at x and Y is a T_1 -space, then the set $f[x]$ coincides with the singleton $\{f(x)\}$.

The following theorem implies that for a compact-preserving function $f : X \rightarrow Y$ from a Fréchet space X to a Hausdorff space Y , the continuity of f at a point $x \in X$ is equivalent to the equality $f[x] = \{f(x)\}$.

Theorem 1. *If $f : X \rightarrow Y$ is a compact-preserving function from a (strong) Fréchet space X to a Hausdorff topological space Y , then for each point $x \in X$ and a neighborhood $O_{f(x)}$ of $f(x)$ in Y there is a neighborhood O_x of x in X such that $f(O_x) \subset f[x] \cup O_{f(x)}$ (and the set $f[x] \setminus O_{f(x)}$ is finite).*

Proof. Assume conversely that there is a neighborhood $O_{f(x)} \subset Y$ of $f(x)$ such that $f(O_x) \not\subset f[x] \cup O_{f(x)}$ for each neighborhood $O_x \subset X$ of x . This means that the set $A = f^{-1}(Y \setminus (f[x] \cup O_{f(x)}))$ contains the point x in its closure. By the Fréchet property of X , the set A contains a sequence $(x_n)_{n \in \omega}$ that converges to the point x . Replacing this sequence by a suitable subsequence we can assume that either the set $\{f(x_n)\}_{n \in \omega}$ is a singleton or else $f(x_n) \neq f(x_m)$ for all $n \neq m$. In the first case the singleton $\{f(x_0)\} = \{f(x_n)\}_{n \in \omega}$ lies in the set $f[x]$, which is not possible as $x_0 \in A \cap f^{-1}(f[x]) = \emptyset$. So, we have the second option which implies that the set $K = \{f(x)\} \cup \{f(x_n)\}_{n \in \omega}$ is infinite. Since the function f is compact-preserving, the set K is compact and being infinite, has an accumulating point $y \in K$, which is not equal to $f(x)$ (as $f(x)$, being a unique point of the intersection $K \cap O_{f(x)}$, is isolated in K). Then the preimage $f^{-1}(y)$ does not contain the point x and hence the set $S = \{x\} \cup \{x_n\}_{n \in \omega} \setminus f^{-1}(y)$ is compact. On the other hand, its image $f(S) = K \setminus \{y\}$ is not compact, which contradicts the compact-preserving property of f .

Next, assuming that the space X is strong Fréchet, we shall prove that the complement $f[x] \setminus O_{f(x)}$ is finite. Assume conversely that this complement contains a sequence $\{y_n\}_{n \in \omega}$ of pairwise distinct points. For every $n \in \omega$ consider the set $A_n = \bigcup_{m \geq n} f^{-1}(y_m)$ and observe that $x \in \bigcap_{n \in \omega} \bar{A}_n$. Since X is strong Fréchet, there is a sequence of points $a_n \in A_n$, $n \in \omega$, which converges to x . The compact-preserving property of f guarantees that the set $K = \{f(x)\} \cup \{f(a_n)\}_{n \in \omega} \subset \{f(x)\} \cup \{y_n\}_{n \in \omega}$ is compact. Being infinite, this set has an accumulation point $y \in K$, which is not equal to the isolated point $f(x)$ of K . Then the set $S = \{x\} \cup \{a_n\}_{n \in \omega} \setminus f^{-1}(y)$ is compact while its image $f(S) = K \setminus \{y\}$ is not. But this contradicts the compact-preserving property of f . \square

Theorem 1 implies the following characterization of compact-preserving functions defined on strong Fréchet spaces.

Corollary 1. *A function $f : X \rightarrow Y$ from a (strong Fréchet) space X to a Hausdorff space Y is compact-preserving if (and only if) for each point $x \in X$ there is a subset $K_x \subset Y$ such that for each neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there is a neighborhood $O_x \subset X$ of x such that $f(O_x) \subset O_{f(x)} \cup K_x$ and $K_x \setminus O_{f(x)}$ is finite.*

Proof. The “only if” part follows from Theorem 1.

To prove the “if” part, assume that for each point $x \in X$ there is a subset $K_x \subset Y$ such that for each neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there is a neighborhood $O_x \subset X$ of x such that $f(O_x) \subset O_{f(x)} \cup K_x$ and $K_x \setminus O_{f(x)}$ is finite. Given a compact subset $K \subset X$ we need to prove that the image $f(K)$ is compact. Let \mathcal{U} be a cover of $f(K)$ by open subsets of Y . For each point $x \in X$ find an open set $O_{f(x)} \in \mathcal{U}$ that contains the point $f(x) \in f(K)$. Next, find an open neighborhood $O_x \subset X$ of x such that $f(O_x) \subset O_{f(x)} \cup K_x$. The finiteness of the set $K_x \setminus O_{f(x)}$ implies that the set $f(O_x) \setminus O_{f(x)}$ is finite.

The compactness of the set K guarantees that the open cover $\{O_x : x \in K\}$ of K contains a finite subcover $\{O_x : x \in F\}$. Choose a finite subfamily $\mathcal{F} \subset \mathcal{U}$ whose union $\bigcup \mathcal{F}$ contains the finite subset $\bigcup_{x \in F} f(K \cap O_x) \setminus O_{f(x)}$ of $f(K) \subset \bigcup \mathcal{U}$. Then $\mathcal{V} = \mathcal{F} \cup \{O_{f(x)} : x \in F\}$ is the required finite subcover of $f(K)$ witnessing that $f(K)$ is compact. \square

Theorem 1 can be applied to give an alternative proof of the mentioned “preserving” characterization of continuous functions. We recall that a function $f : X \rightarrow Y$ has the *Darboux property* if the image $f(C)$ of each connected subset $C \subset X$ is connected.

Corollary 2. *A function $f : X \rightarrow Y$ from a locally connected strong Fréchet space X to a Hausdorff space Y is continuous if and only if f is compact-preserving and has the Darboux property.*

Proof. The “only if” part is trivial. To prove the “if” part, assume that a function $f : X \rightarrow Y$ is compact-preserving and has the Darboux property. Assuming that the function f is discontinuous at some point $x \in X$, and applying Theorem 1, we conclude that the set $f[x]$ contains a point y , distinct from $f(x)$. Since Y is Hausdorff, we can find disjoint open neighborhoods O_y and $O_{f(x)}$ of the points y and $f(x)$ in Y , respectively. By Theorem 1, the complement $F = f[x] \setminus O_{f(x)}$ is finite. Replacing the neighborhood O_y by $O_y \setminus (F \setminus \{y\})$, we can assume that $O_y \cap (f[x] \cup O_{f(x)}) = \{y\}$, which means that y is an isolated point of the space $f[x] \cup O_{f(x)}$.

By Theorem 1, there is a neighborhood $O_x \subset X$ of x such that $f(O_x) \subset f[x] \cup O_{f(x)}$. Since the space X is locally connected, we can choose the neighborhood O_x to be connected. By the Darboux property, the image $f(O_x)$ is connected and contains the point y by the definition of the set $f[x] \ni y$. On the other hand, the point y is isolated in $f(O_x)$, which implies that $f(O_x)$ cannot be connected. This contradiction completes the proof. \square

Remark 1. In fact, the characterization of the continuity given in Corollary 2 holds for any function defined on a locally connected Fréchet Hausdorff space, see [9].

Now we shall study the continuity properties of compact-preserving functions in more details. Let $f : X \rightarrow Y$ be a function between topological spaces. A sequence $(x_n)_{n \in \omega}$ of points of X will be called

- *injective* if $x_n \neq x_m$ for any distinct numbers $n, m \in \omega$;
- *f -injective* if $f(x_n) \neq f(x_m)$ for any distinct numbers $n, m \in \omega$;
- *f -constant* if $\{f(x_n)\}_{n \in \omega}$ is a singleton.

It is clear that each sequence $(x_n)_{n \in \omega}$ in X with finite (resp. infinite) image $\{f(x_n)\}_{n \in \omega}$ contains an f -constant (resp. f -injective) subsequence. Consequently, each sequence in X contains an f -constant or f -injective subsequence.

The following theorem proved in [9, Lemma 2] shows that f -injective convergent sequences do not see the discontinuity of compact-preserving functions.

Theorem 2. *For any compact-preserving function $f : X \rightarrow Y$ from a topological space X to a Hausdorff space Y and each f -injective sequence $\{x_n\}_{n \in \omega} \subset X$ that converges to a point $x \in X$ the sequence $(f(x_n))_{n \in \omega}$ converges to the point $f(x)$.*

Proof. The set $K = \{x\} \cup \{x_n\}_{n \in \omega}$ is compact and so is its image $f(K)$. Since the sequence $(x_n)_{n \in \omega}$ is f -injective, the convergence of $(f(x_n))_{n \in \omega}$ to $f(x)$ will follow as soon as we check that $f(x)$ is a unique non-isolated point of the compact space $f(K)$. Assuming that $y \neq f(x)$ is another non-isolated point of $f(K)$, we observe that the set $C = K \setminus f^{-1}(y)$ is compact but its image $f(C) = f(K) \setminus \{y\}$ is not, which is impossible as f is compact-preserving. \square

Now we establish the promised continuity of a compact-preserving function f on the set SI_f of all points $x \in X$ at which f is sequentially infinite.

Theorem 3. *For each compact-preserving function $f : X \rightarrow Y$ from a Fréchet space X to a Hausdorff space the restriction $f|SI_f$ is continuous.*

Proof. Assume that $f|SI_f$ is discontinuous at some point $x \in SI_f$. In this case we can find a neighborhood $O_{f(x)}$ of $f(x)$ in Y such that the set $A = SI_f \setminus f^{-1}(O_{f(x)})$ contains the point x in its closure. By the Fréchet property of X , there is a sequence $\{x_n\}_{n \in \omega} \subset A$ that converges to the point x . Passing to a suitable subsequence, we can assume that the sequence $(x_n)_{n \in \omega}$ is f -injective or f -constant. Since the sequence $(f(x_n))_{n \in \omega}$ does not converge to $f(x)$, it cannot be f -injective by Theorem 2. So, $(x_n)_{n \in \omega}$ is f -constant and hence $\{f(x_n)\}_{n \in \omega} = \{y\}$ for some $y \in Y$. Since Y is Hausdorff, the points $f(x)$ and y have disjoint open neighborhoods $U_{f(x)} \subset O_{f(x)}$ and $U_y \subset Y$, respectively.

For every $n \in \omega$ the function f is sequentially infinite at x_n . Consequently, there is a convergent to x sequence $(x_{n,m})_{m \in \omega}$ with infinite image $\{f(x_{n,m})\}_{m \in \omega}$. By a standard diagonal argument, we can replace the sequences $(x_{n,m})_{m \in \omega}$, $n \in \omega$, by suitable subsequences, and assume that the double sequence $(x_{n,m})_{n,m \in \omega}$ is f -injective in the sense that $f(x_{i,k}) \neq f(x_{j,m})$ for any distinct pairs $(i,k), (j,m) \in \omega \times \omega$.

By Theorem 2, for every $n \in \omega$ the sequence $(f(x_{n,m}))_{m \in \omega}$ converges to $f(x_n) = y$. Again passing to a subsequence, we can assume that $f(x_{n,m}) \in U_y$ for all $m \in \omega$. Now consider the set $\{x_{n,m}\}_{n,m \in \omega}$ and observe that it contains the set $\{x\} \cup \{x_n\}_{n \in \omega}$ in its closure. Then Fréchet property of X yields an injective sequence $\{z_k\}_{k \in \omega} \subset \{x_{n,m}\}_{n,m \in \omega}$ that converges to x . The f -injectivity of the double sequence $(x_{n,m})_{n,m \in \omega}$ implies the f -injectivity of the sequence $(z_k)_{k \in \omega}$. By Theorem 2, the sequence $\{f(z_k)\}_{k \in \omega} \subset U_y$ converges to $f(x)$, which is not possible as $f(x)$ does not belong to the closure of U_y . \square

In light of Theorem 3 it is important to know the structure of the set SI_f . Let us recall that by LI'_f we denote the (closed) set of non-isolated points of the (closed) set LI_f of points $x \in X$ at which the function f is locally infinite.

A topological space X is called *sequentially Hausdorff* if each convergent sequence in X has a unique limit.

Theorem 4. *For a compact-preserving function $f : X \rightarrow Y$ from a sequentially Hausdorff Fréchet space X to a Hausdorff space Y ,*

- (1) *the set SI_f is closed;*
- (2) *$LI'_f \subset SI_f \subset LI_f$;*
- (3) *$SI_f = LI_f$ provided that X is strong Fréchet.*

Proof. 1. Given any point x in the closure $\overline{SI_f}$ of SI_f , apply the Fréchet property of X and find a sequence $\{x_n\}_{n \in \omega} \subset SI_f$ that converges to x . For every $n \in \omega$ the definition of the set $SI_f \ni x_n$ yields an f -injective sequence $(x_{n,m})_{m \in \omega}$ that converges to x_n . By a standard diagonal inductive procedure, we can replace the sequences $(x_{n,m})_{m \in \omega}$ by suitable subsequences and assume that the double sequence $(x_{n,m})_{n,m \in \omega}$ is f -injective and the infinite set $\{f(x_{n,m})\}_{n,m \in \omega}$ does not contain $f(x)$. Since the set $\{x_{n,m}\}_{n,m \in \omega} \not\ni x$ contains x in its closure, by the Fréchet property of X , there is an injective sequence $(z_k)_{k \in \omega} \subset \{x_{n,m}\}_{n,m \in \omega}$ that converges to x . The f -injectivity of the double sequence $(x_{n,m})_{n,m \in \omega}$ implies the f -injectivity of the sequence $(z_k)_{k \in \omega}$, which witnesses that f is sequentially infinite at x and hence $x \in SI_f$.

2. The inclusion $SI_f \subset LI_f$ trivially follows from the definitions. To prove that $LI'_f \subset SI_f$, fix a non-isolated point x of the set LI_f and using the Fréchet property of X , find a sequence $\{x_n\}_{n \in \omega} \subset LI_f \setminus \{x\}$ that converges to x .

Let $F_{-1} = \{f(x), f(x_0)\}$. By induction, for every $n \in \omega$ we shall construct a finite subset $F_n \subset X$ and a sequence $(x_{n,m})_{m \in \omega}$ convergent to the point x_n such that

- (a) $\{f(x_{n,m})\}_{m \in \omega} \subset Y \setminus F_{n-1}$;
- (b) the sequence $(x_{n,m})_{m \in \omega}$ is either f -injective or f -constant;
- (c) $F_{n-1} \cup \{f(x_{n+1})\} \subset F_n$;
- (d) $\{f(x_{n,m})\}_{m \in \omega} \subset F_n$ if the sequence $(x_{n,m})_{m \in \omega}$ is f -constant.

Assume that for some $n \in \omega$ the finite set F_{n-1} has been constructed. The condition (c) of the inductive construction guarantees that $f(x_n) \in F_{n-1}$. Since f is locally infinite at x_n , the point x_n does not belong to the interior of the set $f^{-1}(F_{n-1})$. Consequently, the set $A_n = X \setminus f^{-1}(F_{n-1})$ contains x_n in its closure. By the Fréchet property of X , there is a sequence $\{x_{n,m}\}_{m \in \omega} \subset A_n$ which converges to x_n . Passing to a subsequence we can additionally assume that this sequence is either f -injective or f -constant. Put $F_{n+1} = F_n \cup \{f(x_{n+1})\}$ if the sequence $(x_{n,m})_{m \in \omega}$ is f -injective and $F_n = F_{n-1} \cup \{f(x_{n+1})\} \cup \{f(x_{n,m}) : m \in \omega\}$, otherwise.

After completing the inductive construction, we obtain an increasing sequence $(F_n)_{n \in \omega}$ of finite subsets of Y and f -injective or f -constant sequences $(x_{n,m})_{m \in \omega}$, $n \in \omega$, with $\lim_{m \rightarrow \infty} x_{n,m} = x_n$. Replacing $(x_n)_{n \in \omega}$ by a suitable subsequence, we can assume that either for all $n \in \omega$ the sequence $(x_{n,m})_{m \in \omega}$ is f -constant or for all $n \in \omega$ this sequence is f -injective.

In the second case we can apply a standard inductive diagonal argument and replacing each sequence $(x_{n,m})_{m \in \omega}$, $n \in \omega$, by a suitable subsequence, assume that the double sequence $(x_{n,m})_{n,m \in \omega}$ is f -injective.

Since the set $\{x_{n,m}\}_{n,m \in \omega}$ contains the set $\{x\} \cup \{x_n\}_{n \in \omega}$ in its closure, the Fréchet property of X yields an injective sequence $\{z_k\}_{k \in \omega} \subset \{x_{n,m}\}_{n,m \in \omega}$ that converges to x .

If for all $n \in \omega$ the sequence $(x_{n,m})_{m \in \omega}$ is f -injective, then so is the double sequence $(x_{n,m})_{n,m \in \omega}$ and its injective subsequence $(z_k)_{k \in \omega}$, which witnesses that f is sequentially infinite at x .

Now assume that for all $n \in \omega$ the sequence $(x_{n,m})_{m \in \omega}$ is f -constant. Since the space X is sequentially Hausdorff and $(z_k)_{k \in \omega}$ converges to $x \neq x_n$, $n \in \omega$, the set $\{n \in \omega : \{z_k\}_{k \in \omega} \cap \{x_{n,m}\}_{m \in \omega} \neq \emptyset\}$ is infinite. By the construction, for any distinct numbers $n, k \in \omega$ the singletons $\{f(x_{n,m})\}_{m \in \omega}$ and $\{f(x_{k,m})\}_{m \in \omega}$ are

distinct, which implies that the set $\{f(z_k)\}_{k \in \omega}$ is infinite and hence the sequence $(z_k)_{k \in \omega}$ witnesses that the function f is sequentially infinite at x . So, $x \in SI_f$.

3. Assuming that X is strong Fréchet, we shall prove that each point $x \in LI_f$ belongs to SI_f . Assume conversely that $x \notin SI_f$. Let $F_{-1} = \{f(x)\}$. By induction, for every $n \in \omega$ we shall construct a finite subset $F_n \subset X$ and an f -constant sequence $(x_{n,m})_{m \in \omega}$ convergent to the point x such that

- (1) $\{f(x_{n,m})\}_{m \in \omega} \subset Y \setminus F_{n-1}$;
- (2) $F_n = F_{n-1} \cup \{f(x_{n,m})\}_{m \in \omega}$.

Assume that for some $n \in \omega$ the finite set F_{n-1} has been constructed. Since f is locally infinite at x , the point x does not belong to the interior of the set $f^{-1}(F_{n-1})$. Consequently, the set $X_n = X \setminus f^{-1}(F_{n-1})$ contains x in its closure. By the Fréchet property, there is a sequence $\{x_{n,m}\}_{m \in \omega} \subset X_n$ which converges to x . Moreover, we can assume that this sequence is f -injective or f -constant. Since $x \notin SI_f$, the sequence $\{x_{n,m}\}_{m \in \omega}$ is f -constant and we can put $F_n = F_{n-1} \cup \{f(x_{n,m}) : m \in \omega\}$.

After completing the inductive construction, we obtain an increasing sequence $(F_n)_{n \in \omega}$ of finite subsets of Y and a sequence $(x_{n,m})_{m \in \omega}$, $n \in \omega$, of f -constant sequences that converge to x . For every $n \in \omega$ put $A_n = \{x_{k,m} : k, m \in \omega, k \geq n\}$. Since $x \in \bigcap_{n \in \omega} \bar{A}_n$, the strong Fréchet property of X yields a sequence of points $a_n \in A_n$, $n \in \omega$, which converges to x . Since $f(a_n) \in \bigcup_{n \in \omega} F_n \setminus \bigcup_{k=0}^{n-1} F_k$, the set $\{f(a_n)\}_{n \in \omega}$ is infinite, witnessing that the function f is sequentially infinite at x and hence $x \in SI_f$. \square

Theorems 3 and 4 imply:

Corollary 3. *Let $f : X \rightarrow Y$ be a compact-preserving function from a sequentially Hausdorff space X to a Hausdorff space Y .*

- (1) *If X is Fréchet, then the restriction $f|LI'_f$ is continuous.*
- (2) *If X is strong Fréchet, then the restriction $f|LI_f$ is continuous.*

Corollary 4. *For each compact-preserving function $f : X \rightarrow Y$ from a sequentially Hausdorff Fréchet space X to a Hausdorff space Y there is a point $x \in X$ at which f is locally finite or continuous.*

Proof. If there exists a point $x \in X \setminus LI_f$, then f is locally finite at x . So, we assume that $LI_f = X$. If the space $X = LI_f$ has an isolated point $x \in X$, then f is continuous at x . If $X = LI_f$ has no isolated points, then $LI'_f = LI_f = X$ and the function $f = f|LI'_f$ is continuous according to Corollary 3(1). \square

Now we present two examples showing that Theorem 3, 4 and Corollary 3 cannot be improved.

Example 1. *There is a function $f : X \rightarrow \mathbb{R}$ from a Hausdorff Fréchet countable space X such that $SI_f = LI'_f \neq LI_f$ and $f|LI_f$ is discontinuous.*

Proof. Consider the space $X = \omega^0 \cup \omega^1 \cup \omega^3$ endowed with the topology τ in which

- (a) each point $(k, n, m) \in \omega^3 \subset X$ is isolated;
- (b) a set $U \subset X$ is a neighborhood of a point $(k) \in \omega^1 \subset X$ if and only if $(k) \in U$ and for every $n \in \omega$ there a number $m_n \in \omega$ such that $(k, n, m) \in U$ for all $m \geq m_n$;
- (c) a set $V \subset X$ is a neighborhood of the point $\emptyset \in \omega^0 \subset X$ if and only if $\emptyset \in V$ and there is a number $k_0 \in \omega$ such that for all $k \geq k_0$ and $n, m \in \omega$ we get $(k) \in V$ and $(k, n, m) \in V$.

It is easy to see that the space X is Fréchet but not strong Fréchet.

Now define a function $f : X \rightarrow \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 0 & \text{if } x = \emptyset \in \omega^0 \\ 1 & \text{if } x \in \omega^1 \\ 2^{-(k+n)} & \text{if } x = (k, n, m) \in \omega^3 \end{cases}$$

It is easy to check that the function is compact-preserving, $LI_f = \omega^0 \cup \omega^1$, $SI_f = LI'_f = \omega^0$, and $f|LI_f$ is discontinuous at the point $\emptyset \in \omega^0$. \square

A topological space X is called a s_ω -space if there is a countable family \mathcal{K} of compact metrizable subsets of X generating the topology of X in the sense that a set $U \subset X$ is open if and only if for each compact set $K \in \mathcal{K}$ the intersection $U \cap K$ is open in K . It is well-known that s_ω -spaces are sequential and normal (even stratifiable).

Example 2. *There is a countable s_ω -space X and a compact-preserving function $f : X \rightarrow \mathbb{R}$ which is locally infinite and discontinuous at each point $x \in X$.*

Proof. Consider the space $X = \bigcup_{n \in \omega} \omega^n$ of all finite sequences of natural numbers, endowed with the topology τ in which a subset $U \subset X$ is a neighborhood of a point $s = (s_0, \dots, s_{n-1}) \in \omega^n \subset X$ if and only if $s \in U$ and there exists a number $m_0 \in \omega$ such that for each $m \geq m_0$ the point $s \hat{\ } m = (s_0, \dots, s_{n-1}, m)$ belongs to U . It follows that the sequence $(s \hat{\ } m)_{m \in \omega}$ converges to s and hence the set $K_s = \{s\} \cup \{s \hat{\ } m\}_{m \in \omega}$ is compact. It is easy to see that X is a countable s_ω -space whose topology is generated by the countable family $\mathcal{K} = \{K_s : s \in X\}$ of compact subsets of X . The space X was first described by Arhangel'skii and Franklin [2] and is known in General Topology as the *the Arhangel'skii-Franklin space*.

Now consider the function $f : X \rightarrow \mathbb{R}$ assigning to each sequence $s = (s_0, \dots, s_{n-1}) \in \omega^n \subset X$ the real number

$$f(s) = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n+2^{-s_{n-1}} & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that the function f is compact-preserving. On the other hand, each non-empty open set $U \subset X$ has unbounded image $f(U)$ in \mathbb{R} , which implies that f locally infinite and discontinuous at each point $x \in X$. \square

Remark 2. In the proof of our results about compact-preserving functions we did not use the full strength of the compact-preserving property. What we actually used was the compactness of the images of compact subsets with a unique non-isolated point. On the other hand, by transfinite induction it is easy to construct a function $f : [0, 1] \rightarrow [0, 1]$ which maps each uncountable compact subset of $[0, 1]$ onto $[0, 1]$. Such a function f preserves the compactness of uncountable compact sets and has the Darboux property but is everywhere discontinuous.

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